Math 72

Extra Problems 8.1–8.4

Extra examples solving quadratic equations by completing the square

Example: find value of k to make a perfect square

Example: work rate → quadratic formula

Example: geometry → quadratic formula

Examples: Using complete the square to write a quadratic function in vertex form

$f(x) = a(x-h)^2 + k$

& graphing a quadratic function $a \neq 1$

Where does the vertex formula come from?

using complete the square on $ax^2+bx+c=0$.

Extra Practice Problems

(1) A holding pen for cattle is square with diagonal 100 m.

a) Find the lengths of the sides.

b) Find the area of the pen.

(2) A window's length is 7.3 inches longer than its width. Its area is 569.9 sq. Inches. Find the dimensions.

(3) The base of a triangle is 4 more than twice its height. If the area of the triangle is 42 cm^2, find its base & height.

(4) The number of visitors to U.S. theme parks can be modeled by $V(x) = 0.25x^2 + 2.6x + 315.6$ where $V(x)$ is the number of visitors (in millions) and x is the number of years after 2000.

a) Find the number of visitors is 2005, to nearest million.

b) Project the number of visitors in 2010, to nearest million.

Solve by CTS

(5) $2y^2 - 10y + 11 = 0$ $\qquad y = \frac{5}{2} \pm \frac{\sqrt{3}}{2}$

(6) $3x^2 - x - 52 = 0$ $\qquad x = -4, \frac{13}{3}$

(7) $9x^2 - 18x = -11$ $\qquad x = 1 \pm \frac{i\sqrt{2}}{3}$

(8) $3y^2 - 4y - 1 = 0$ $\qquad y = \frac{2}{3} \pm \frac{\sqrt{7}}{3}$

(9) $3x^2 + 2 = 0$ $\qquad x = \pm \frac{i\sqrt{6}}{3}$

(10) An object thrown upward from the top of a 200ft cliff with initial velocity 12 ft per sec. The height h (from the base of the cliff) after t seconds is

$$h = -16t^2 + 12t + 200$$

When will it hit the ground? Round to the nearest tenth of a second.

$t = 2.4$ sec

From graph, determine how many solutions and their type

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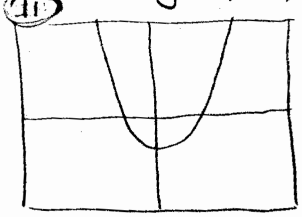
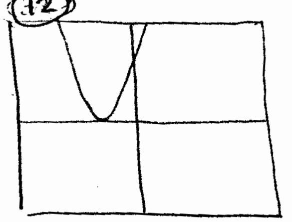
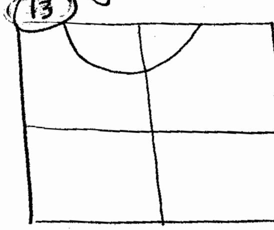
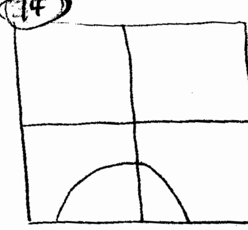
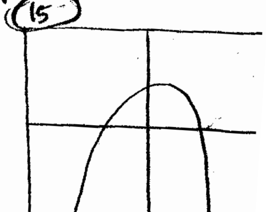

2 real

(12)

probably 1 real

(13)

2 complex

(14)

2 complex

(15)

2 real

Method 1a: CTS by Factoring out leading coefficient

$3x^2 - 9x + 8 = 0$ (2 complex)

$3(x^2 - 3x \qquad) = -8$

$\#^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4}$

$3(x^2 - 3x + \frac{9}{4}) = -8 + 3 \cdot \frac{9}{4}$

$3(x - \frac{3}{2})^2 = \frac{-5}{4}$

HINT: Use GC and >frac

$(x - \frac{3}{2})^2 = \frac{-5}{12}$

$x - \frac{3}{2} = \pm\sqrt{\frac{-5}{12}}$

$x = \frac{3}{2} \pm \frac{i\sqrt{5} \cdot \sqrt{3}}{2\sqrt{3} \cdot \sqrt{3}}$

$\boxed{x = \frac{3}{2} \pm \frac{i\sqrt{15}}{6}}$

$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(3)(8)}}{2(3)}$

$x = \frac{9}{6} \pm \frac{\sqrt{-15}}{6}$

$x = \frac{3}{2} \pm \frac{i\sqrt{15}}{6}$ 2 complex solutions

Method 1B
CTS: divide by leading coef.

$3x^2 - 9x + 8 = 0$ (2 complex)

$\frac{3x^2}{3} - \frac{9x}{3} = \frac{-8}{3}$

$x^2 - 3x = \frac{-8}{3}$

CTS $\begin{cases} \# = \frac{-3}{2} & \text{← use in factor} \\ \#^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4} & \text{← add to both sides} \end{cases}$

$x^2 - 3x + \frac{9}{4} = \frac{-8}{3} + \frac{9}{4}$

MATH > frac is your friend!!

$(x - \frac{3}{2})^2 = \frac{-5}{12}$

Simplify $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

$x - \frac{3}{2} = \frac{\pm i\sqrt{5}}{2\sqrt{3}}$

$x = \frac{3}{2} \pm \frac{i\sqrt{5} \cdot \sqrt{3}}{2\sqrt{3} \cdot \sqrt{3}} \Rightarrow \boxed{x = \frac{3}{2} \pm \frac{i\sqrt{15}}{6}}$

2 complex solutions

$6x^2 - 17x - 14 = 0$ (2 rational) Method 1A: CTS factor out leading coef

$6(x^2 - \frac{17}{6}x \quad) = 14$

Use GC > frac

$\#^2 = \left(\frac{-17}{6} \cdot \frac{1}{2}\right)^2 = \frac{289}{144}$

$6\left(x^2 - \frac{17}{6}x + \frac{289}{144}\right) = 14 + 6 \cdot \frac{289}{144}$

$6\left(x - \frac{17}{12}\right)^2 = \frac{625}{24}$

$\left(x - \frac{17}{12}\right)^2 = \frac{625}{24(6)}$

$x - \frac{17}{12} = \pm\sqrt{\frac{625}{144}}$

$x = \frac{17}{12} \pm \frac{25}{12}$

$x = \frac{17}{12} + \frac{25}{12}, \frac{17}{12} - \frac{25}{12}$

$x = \frac{42}{12}, \frac{-8}{12}$

$\boxed{x = \frac{7}{2}, \frac{-2}{3}}$

2 real rational solutions!
Could be factored!
$(2x-7)(3x+2) = 0$

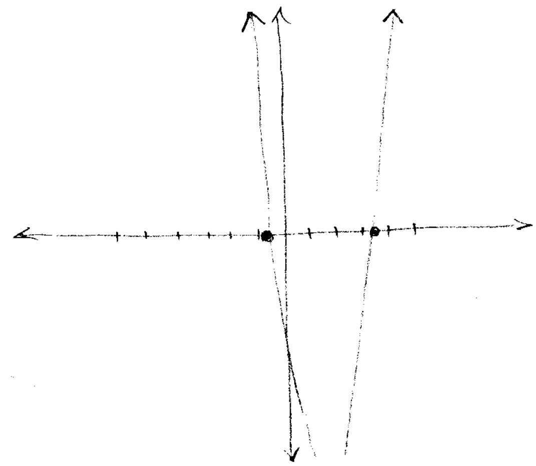
$x = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(6)(-14)}}{2(6)}$

$x = \frac{17 \pm \sqrt{289 + 336}}{12}$

$x = \frac{17}{12} \pm \frac{\sqrt{625}}{12}$

$x = \frac{17}{12} \pm \frac{25}{12}$

$\boxed{x = \frac{7}{2}, \frac{-2}{3}} = 3.5, -.\bar{6}$



$6x^2 - 17x - 14 = 0$ (2 rational) Method 1B: CTS divide by leading coef

$\frac{6x^2}{6} - \frac{17}{6}x = \frac{14}{6}$

$x^2 - \frac{17}{6}x = \frac{7}{3}$

CTS $\begin{cases} \# = -\frac{17}{6} \cdot \frac{1}{2} = \frac{-17}{12} & \leftarrow \text{factor} \\ \#^2 = \left(-\frac{17}{12}\right)^2 = \frac{289}{144} & \leftarrow \text{add to both sides} \end{cases}$

$x^2 - \frac{17}{6}x + \frac{289}{64} = \frac{7}{3} + \frac{289}{144}$

$\boxed{\text{MATH}}$ > frac is your friend!

$\left(x - \frac{17}{12}\right)^2 = \frac{625}{144}$

$x - \frac{17}{12} = \pm\frac{\sqrt{625}}{\sqrt{144}}$

$x = \frac{17}{12} \pm \frac{25}{12}$

$x = \frac{17}{12} + \frac{25}{12} = \frac{42}{12} = \boxed{\frac{7}{2}}$

$x = \frac{17}{12} - \frac{25}{12} = -\frac{8}{12} = \boxed{-\frac{2}{3}}$

23) $25x^2 - 20x + 4 = 0$ (1 rational)

$25\left(x^2 - \frac{20}{25}x \qquad\right) = -4$

$25\left(x^2 - \frac{4}{5}x \qquad\right) = -4$

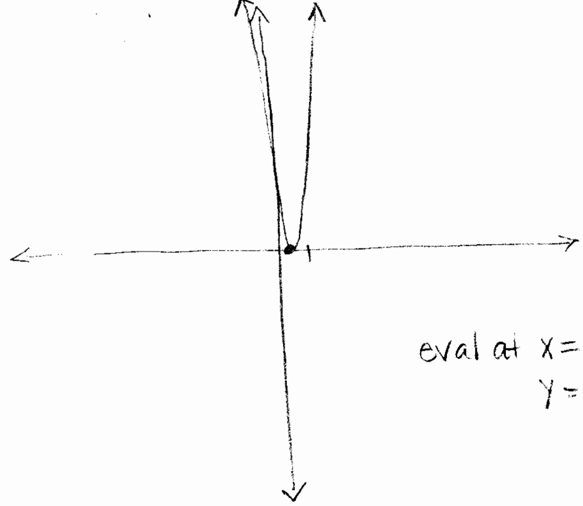
$\#^2 = \left(\frac{-4}{5} \cdot \frac{1}{2}\right)^2 = \left(\frac{-2}{5}\right)^2 = \frac{4}{25}$

$25\left(x^2 - \frac{4}{5}x + \frac{4}{25}\right) = -4 + 25\left(\frac{4}{25}\right)$

$25\left(x - \frac{2}{5}\right)^2 = 0$

$\left(x - \frac{2}{5}\right)^2 = 0$

$\boxed{x = \frac{2}{5}}$ ← one real rational solution



eval at $x = .4$
$y = 0$.

Note: This can be factored to a better perfect square in the first step:
$(5x-2)^2 = 0$.

$25x^2 - 20x + 4 = 0$ (1 rational)

$\frac{25x^2}{25} - \frac{20x}{25} = \frac{-4}{25}$

$x^2 - \frac{4}{5}x \qquad = \frac{-4}{25}$

$\# = \frac{-4}{5} \cdot \frac{1}{2} = \frac{-4}{10} = \frac{-2}{5}$ ← factor

$\#^2 = \frac{4}{25}$ ← add to both sides

$x^2 - \frac{4}{5}x + \frac{4}{25} = \frac{-4}{25} + \frac{4}{25}$

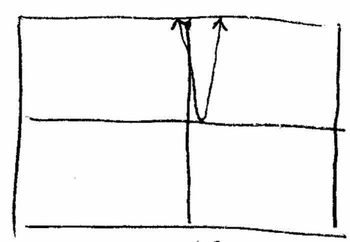
$\left(x - \frac{2}{5}\right)^2 = 0$

$x - \frac{2}{5} = \pm\sqrt{0}$

$x - \frac{2}{5} = 0$

$\boxed{x = \frac{2}{5}}$

Graph $y_1 = 25x^2 - 20x + 4$



vertex at $\left(\frac{2}{5}, 0\right)$

"Zero" will not correctly calculate values when graph does it cross over x-axis ☹

MG
8.1.105 Find a value of k that will make $x^2 + kx + 9$ a perfect square trinomial.

$k = \boxed{3}$

Write an equation to find k.

We know the CTS procedure:

Take coefficient of x and divide by 2:

$$\frac{k}{2}$$

Square result

$$\left(\frac{k}{2}\right)^2 = \frac{k^2}{4}$$

In the given problem this result is 9:

$$\frac{k^2}{4} = 9$$

Solve for k:

$$k^2 = 9 \cdot 4$$

$$k^2 = 36$$

$$k = 6 \quad \text{or} \quad k = -6.$$

Notice the instructions ask for <u>a</u> value of k.

MathXL is prepared for one answer.

It will count

$k=6$ correct

or

$k=-6$ correct

but not

$k = 6, -6.$

MG

8.3.63 A father and his son can clean the house together in 5 hours. When the son works alone, it takes him an hour longer to clean than it takes his dad alone. Find how long it takes the son to clean alone.

It takes the son ☐ hours to clean alone. (Round to the nearest tenth.)

dad = x hours to do entire job

Son = $x+1$ hours

fractions done in 1 hour:

$$\frac{1}{x} + \frac{1}{x+1} = \frac{1}{5}$$

LCD = $5x(x+1)$

$$5(x+1) + 5x = x(x+1)$$

dist

$$5x + 5 + 5x = x^2 + x$$

$$10x + 5 = x^2 + x$$

$$0 = x^2 - 9x - 5$$

QF

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-5)}}{2}$$

$$x = \frac{9 \pm \sqrt{81 + 20}}{2}$$

$$x = \frac{9 \pm \sqrt{101}}{2}$$

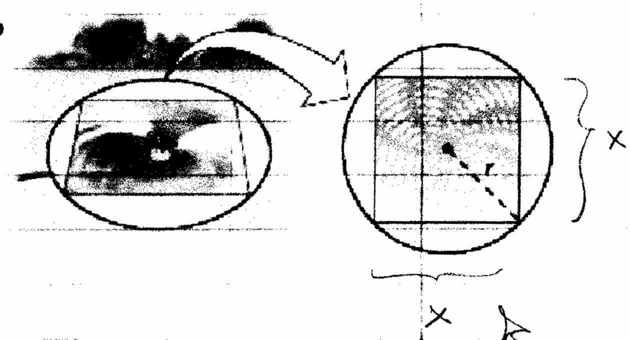
$x \approx 9.5$ or $-.5$ discard negative time

x = dad

Question asks for son.

$9.5 + 1 = \boxed{10.5 \text{ hrs}}$

MG 8.3.71 A sprinkler that sprays water in a circular motion is to be used to water a square garden. If the area of the garden is 720 square feet, find the smallest whole number radius that the sprinkler can be adjusted to so that the entire garden is watered.



The smallest whole number radius is ☐ feet.

This is an A+ question! You have to remember some geometry. They tell us the area of the square, but that relates to x, the dimension of the square.

Step 1 Find x: $x^2 = 720$ ← area of square

$x = \pm\sqrt{720}$

negative is extraneous

$x = 12\sqrt{5}$

720 → 72, 10; 72 → 8, 9; 10 → 2, 5; 8 → 4, 2; 9 → 3, 3; 4 → 2, 2

$720 = 2^4 \cdot 3^2 \cdot 5$

so

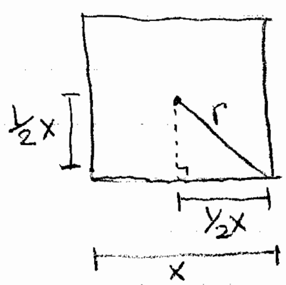
$\sqrt{720} = 2^2 \cdot 3\sqrt{5}$

$= 12\sqrt{5}$

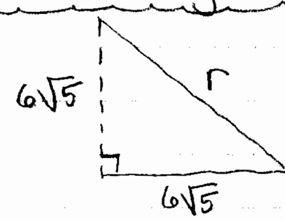
But x is <u>NOT</u> r!

$\frac{1}{2}x \Rightarrow \frac{1}{2}(12\sqrt{5})$

$\frac{1}{2}x \Rightarrow 6\sqrt{5}$



Step 2: Draw a right Δ: Notice that both legs are $\frac{1}{2}x = 6\sqrt{5}$



Step 3: Use Pythagorean Theorem to find r.

$a^2 + b^2 = c^2$

$(6\sqrt{5})^2 + (6\sqrt{5})^2 = r^2$

$36 \cdot 5 + 36 \cdot 5 = r^2$

$72 \cdot 5 = r^2$

$360 = r^2$

$\sqrt{360} = r$

step 4: calculate r and use next higher whole # for answer.

$r = \sqrt{360} \approx 18.9$

$\boxed{r = 19 \text{ feet}}$

Review

① Rewrite $f(x) = -3x^2 - 12x - 11$
a) using CTS
b) using vertex formula.

a) $y = -3x^2 - 12x - 11$

$y + 11 = -3(x^2 + 4x \quad)$

$\# = \frac{4}{2} = 2 \leftarrow$ goes in squared factor later

$\#^2 = 2^2 = 4 \leftarrow$ add inside ()

$y + 11 - 3(4) = -3(x^2 + 4x + 4)$

$y + 11 - 12 = -3(x+2)^2$

$y - 1 = -3(x+2)^2$

$y = -3(x+2)^2 + 1$

$\boxed{f(x) = -3(x+2)^2 + 1}$

b) $h = \frac{-b}{2a} = \frac{-(-12)}{2(-3)} = \frac{12}{-6} = -2$

$k = f(-2) = -3(-2)^2 - 12(-2) - 11$
$= -3(4) + 24 - 11$
$= 1$

$f(x) = a(x-h)^2 + k$

$\boxed{f(x) = -3(x+2)^2 + 1}$

② Solve $-3x^2 - 12x - 11 = 0$.
Compared to ①, what have we found? $\boxed{\text{set } y=0 \text{ means } x\text{-ints}}$

<u>Method 1:</u> Quadratic formula

$a = -3 \quad b = -12 \quad c = -11$

easier $3x^2 + 12x + 11 = 0$

$a = 3 \quad b = 12 \quad c = 11$

(2) con't

$$x = \frac{-12 \pm \sqrt{12^2 - 4(3)(11)}}{2(3)}$$

$$x = \frac{-12 \pm \sqrt{12}}{6}$$

$$x = \frac{-12}{6} \pm \frac{2\sqrt{3}}{6}$$

$$\boxed{x = -2 \pm \frac{\sqrt{3}}{3}}$$

← These are real numbers. The quadratic function has 2 (decimal) x-intercepts.

Method 2: Completing the square.

$$\frac{-3x^2}{-3} \frac{-12x}{-3} \frac{-11}{-3} = \frac{0}{-3}$$

$$x^2 + 4x + \frac{11}{3} = 0$$

$$x^2 + 4x = \frac{-11}{3}$$

$\# = \frac{4}{2} = 2$ ← use in squared factor later

$\#^2 = 2^2 = 4$ ← add to both sides now

$$x^2 + 4x + 4 = \frac{-11}{3} + 4$$

$$(x+2)^2 = \frac{1}{3}$$

$$(x+2) = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}\cdot\sqrt{3}}\cdot\sqrt{3} = \pm\frac{\sqrt{3}}{3}$$

$$\boxed{x = -2 \pm \frac{\sqrt{3}}{3}}$$

Review

(3) Graph $f(x) = -\frac{1}{2}x^2 + 3x - \frac{5}{2}$

vertex $h = \frac{-b}{2a} = \frac{-3}{2\left(-\frac{1}{2}\right)} = \frac{-3}{-1} = 3$

$k = f(3) = -\frac{1}{2}(3)^2 + 3(3) - \frac{5}{2}$

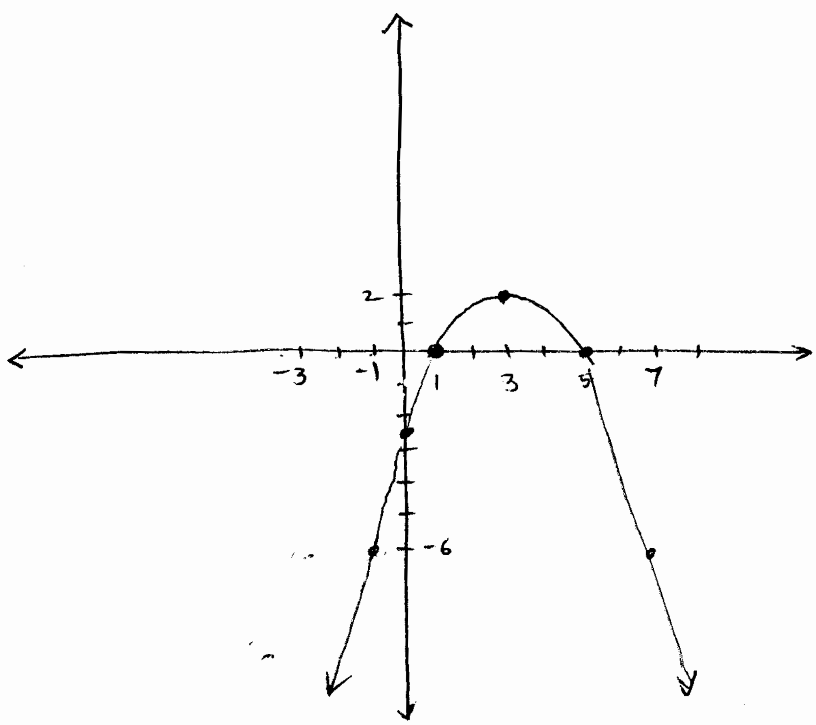
$= 2$

vertex $(3, 2)$

$a = -\frac{1}{2}$ opens downward, wider than standard.

Use GC table for graphing

x	f(x)	
-2	-10.5	
-	-6	✓
0	-2.5	
1	0	✓
2	1.5	
3	2	✓
4	1.5	
5	0	✓
6	-2.5	
7	-6	✓
8	-10.5	



Either change y-scl to 0.5 on graph
or choose points marked ✓ in the table.

Review. Simplify:

(11) $(x-2)^2$

$= (x-2)(x-2)$

$= x^2 - 2x - 2x + 4$

$= \boxed{x^2 - 4x + 4}$

(-2) times 2 $\quad (-2)^2$

(12) $(x+3)^2$

$= (x+3)(x+3)$

$= x^2 + 3x + 3x + 9$

$= \boxed{x^2 + 6x + 9}$

(3) times 2 $\quad (3)^2$

(13) $(x-4)^2$

$= (x-4)(x-4)$

$= x^2 - 4x - 4x + 16$

$= \boxed{x^2 - 8x + 16}$

(-4) times 2 $\quad (-4)^2$

(14) $(x+5)^2$

$= (x+5)(x+5)$

$= x^2 + 5x + 5x + 25$

$= \boxed{x^2 + 10x + 25}$

(5) times 2 $\quad (5)^2$

Notice:

1) The middle term is always
   - The same sign as original middle
   - 2 times the number in second term

2) The last term is always
   - the number in the the second term, squared.

To un-do this

→ divide the middle # by 2

→ square the result

(15) Use completing the square to find the vertex form of the function $f(x) = x^2 - 6x + 11$.

$f(x) = x^2 - 6x + ? - ? + 11$

This one is easier because $a = 1$

want to add, then subtract, a number that will make a perfect square from $x^2 - 6x$

Remember that $-6x$ is the middle term of a perfect square.

→ divide its coefficient by 2 : $\frac{-6}{2} = -3$ *

→ square the result : $(-3)^2 = 9$

Add and subtract this result.

$$f(x) = \underbrace{x^2 - 6x + 9}_{\text{perfect square}} \underbrace{-9 + 11}_{\text{combine like terms}}$$

$$\boxed{f(x) = (x-3)^2 + 2}$$

* The number here is the number we got after dividing by 2

CAUTION: These steps depend on $a = 1$.
If $a \neq 1$, we have more work.

NO 16 Use completing the square to find the vertex form of the function

$f(x) = x^2+4x+1$

$= x^2+4x+ ? - ? +1$ ← This one is easier because $a=1$.

want to add, then subtract, a number that will make a perfect square from x^2+4x

Remember that $4x$ is the middle term of a perfect square.

→ divide its coefficient by 2: $\frac{4}{2}=2$ *

→ square the result: $2^2=4$

Add and subtract this result.

$f(x) = \underbrace{x^2+4x+4}_{\text{we've created a perfect square}} \underbrace{-4+1}$

we've created a perfect square
"completed the square"

$\boxed{f(x) = (x+2)^2 - 3}$

* The number here is the number we got after dividing by 2

Why bother with completing the square?

a) Because it will give us another way to solve quadratic equations that don't factor

b) When we get to circles in chapter 10, we won't have a "center formula" like our "vertex formula". Completing the square will be our only method.

17) Use completing the square to find the vertex form of the function.

$f(x) = 2x^2 - 4x + 5$

Remember, completing the square works only when $a=1$!!
Factor out $a=2$ from first two terms.

$$f(x) = 2(x^2 - 2x + ? - ?) + 5$$

IMPORTANT: want to add and subtract our number inside the parentheses, where $a=1$.

$-2x$ is the middle term of the perfect square

- divide by 2 $\quad \frac{-2}{2} = -1 \quad$ * use in factor
- square result $\quad (-1)^2 = 1$

$$f(x) = 2(\underbrace{x^2 - 2x + 1}_{\text{This part is the perfect square.}} \underbrace{-1}_{\text{This term is inside parentheses. Distribute } a=2 \text{ to this term only}}) + 5$$

$$= 2\underbrace{(x^2 - 2x + 1)}_{\text{perfect square * use } -1} \underbrace{- 2 \cdot 1 + 5}_{\text{arithmetic}}$$

$$\boxed{f(x) = 2(x-1)^2 + 3}$$

(18) Use completing the square to find the vertex form of the function

$$f(x) = \underbrace{-3x^2 + 24x}\, - 46$$

factor out $a = -3$ from these two terms only

$$f(x) = -3(x^2 - 8x + ? - ?) - 46$$

divide by 2: $\frac{-8}{2} = -4$ *

square result: $(-4)^2 = 16$

add and subtract inside parentheses

$$f(x) = -3(\underbrace{x^2 - 8x + 16}_{\text{square}} \underbrace{- 16}_{\substack{\text{dist } -3 \\ \text{to this term only}}}) - 46$$

$$f(x) = -3(\underbrace{x^2 - 8x + 16}_{\text{square}}) - 3(-16) - 46$$

$$f(x) = -3(x-4)^2 + 48 - 46$$

* use result from divide by 2

$$\boxed{f(x) = -3(x-4)^2 + 2}$$

Where does the vertex formula come from?

Use $f(x) = ax^2 + bx + c$ and complete the square.

$$f(x) = a\left(x^2 + \frac{b}{a}x \qquad \right) + c$$ factor out a.

$$\# = \frac{b}{a} \cdot \frac{1}{2} = \frac{b}{2a}$$

$$\#^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$a \cdot \#^2 = a \cdot \frac{b^2}{4a^2} = \frac{b^2}{4a}$$

complete the square

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$f(x) = a\left(x + \underbrace{\frac{b}{2a}}_{x}\right)^2 + \underbrace{\left(c - \frac{b^2}{4a}\right)}_{k}$$

Remember: subtract h
$y = a(x-h)^2 + k$

vertex formula $x = \dfrac{-b}{2a}$

$$y = c - \frac{b^2}{4a}$$

$$y = f\left(\frac{-b}{2a}\right) = a\left(\underbrace{\frac{-b}{2a} + \frac{b}{2a}}_{0}\right)^2 + c - \frac{b^2}{4a}$$

$$y = c - \frac{b^2}{4a}$$

This gives a formula for k also, but most people can't remember it and just plug $\frac{-b}{2a}$ in to get y.